ON THE RESIDUAL FINITENESS OF GENERALIZED FREE PRODUCTS[1]

BY
JOAN LANDMAN DYER

In this paper we shall be concerned with the behavior of residually finite groups under the formation of the generalized free product with one subgroup amalgamated.

A first result in this direction is due to Gruenberg [3], who proved that the free product of residually finite ($R\mathscr{F}$) groups is again $R\mathscr{F}$. Baumslag began the corresponding investigation of the generalized free product (g.f.p.) [1]. He has established, firstly, that the g.f.p. of $R\mathscr{F}$ groups is always $R\mathscr{F}$ under the proviso that the amalgamated subgroup be finite ($\mathscr{F}$), or, in the notation of that paper:

Theorem 1 (Baumslag [1]). *$\sigma(A, B; \mathscr{F}) \subset R\mathscr{F}$ for $A, B \in R\mathscr{F}$ [$\sigma(A, B)$ denoting the set of all g.f.p. of A and B with one amalgamated subgroup, and $\sigma(A, B; \Gamma)$ that subset of $\sigma(A, B)$ in which the amalgamated subgroup satisfies the condition Γ].*

At this point we impose the fairly reasonable condition that all groups involved be finitely generated (f.g.). For f.g. abelian groups ($\mathscr{A}$), the g.f.p. is again always $R\mathscr{F}$ [1]. Moving slowly from the abelian situation, "nice" behavior is no longer the rule, even for groups which are nilpotent of class 2 [1]. Nonetheless, Baumslag does obtain a pleasant description of the structure of $\sigma(A, B)$ for A, B f.g. torsion-free nilpotent, viz.,

Theorem 2 (Baumslag [1]). *If A, B are f.g. torsion-free nilpotent, then*

$$\sigma(A, B) \subset \Phi \cdot R\mathscr{F}$$

and $\sigma(A, B;$ closed in A and $B) \subset R\mathscr{F}$ (where Φ is the class of free groups).

It seems reasonable to suppose that the same result obtains without the requirement that the groups involved be torsion-free. However, somewhat surprisingly, we shall show that this is not the case.

Theorem 3. *There exist f.g. nilpotent groups A, B for which $\sigma(A, B) \not\subset \Phi \cdot R\mathscr{F}$. In fact, $A \cong B$, and nilpotent of class 3.*

However, we can still obtain a description of the g.f.p. as follows:

Theorem 4. *For A, B f.g. nilpotent,*

$$\sigma(A, B) \subset R\mathscr{F} \cdot \Phi \cdot R\mathscr{F}$$

and $\sigma(A, B;$ closed$) \subset \mathscr{F} \cdot R\mathscr{F}$.

Received by the editors March 1, 1966 and, in revised form April 3, 1967.

[1] This work was partially supported by the National Science Foundation under contract NSF GP-5303.

It is an open question whether this result can be improved to: $\sigma(A, B) \subset R\mathscr{F} \cdot R\mathscr{F}$, as I suspect.

Continuing in this direction, the next reasonable class to consider appears to be the polycyclic groups. However, the results above rely on a description of the manner in which normal subgroups of a f.g. torsion-free nilpotent group intersect an arbitrary subgroup, and no such information about the structure of polycyclic groups is known as yet.

In the course of establishing the fact that $\sigma(A, B) \not\subset R\mathscr{F}$ for A, B f.g. torsion-free nilpotent and nonabelian, Baumslag shows that $\sigma(A, B)$ contains a group which contains a non-Hopf group, and conjectures that $\sigma(A, B)$ itself always contains a non-Hopf group. We provide additional evidence for this conjecture below; before stating our result, however, some notation is required. Let

$$1 \to M \to A \to A/M \to 1$$

be an extension of M by A/M. Call A *strongly noncentral* if there exist $m \in M$, $a \in A$ with $gp\{m, m^a\}$ noncyclic ($m^a = a^{-1}ma$). Then

THEOREM 5. *$\sigma(A, B)$ contains a non-Hopf group whenever A, B are any split, strongly noncentral extensions of f.g. torsion-free abelian groups whose centralizers are of finite index.*

For torsion-free nilpotent groups, noncentral extensions are strongly noncentral, as $m^{ra} = m^s$, r, s integral, is possible only if $r = s$. Thus we have the obvious

COROLLARY 1. *$\sigma(A, B)$ contains a non-Hopf group for A, B torsion-free nilpotent and representable as split noncentral extensions of abelian groups whose centralizers are of finite index.*

An easy application of the construction of Theorem 5 yields also:

COROLLARY 2. *$\sigma(A, B)$ contains a non-Hopf group whenever A, B are split extensions of f.g. torsion-free noncyclic abelian groups of rank 1 as A, B modules respectively.*

COROLLARY 3. *$\sigma(A, B)$ contains a non-Hopf group whenever A and B have the form $X \wr Y$ with X abelian containing an element of infinite order, and Y of order at least 2.*

Since Theorem 5 covers groups of abelian-by-finite type, it is pleasantly surprising that

THEOREM 6. *$\sigma(A, B) \subset R\mathscr{F}$ for $A, B \in \mathscr{F} \cdot \mathscr{A}$ (Recall: $\mathscr{A}$ is the class of f.g. abelian groups).*

The proposition which furnishes the key to Theorem 6 may be exploited in several ways:

THEOREM 7. *If $A, B \in \mathscr{A} \cdot \mathscr{F}$, then $\sigma(A, B) \subset R\mathscr{F}$ if and only if at least one of A, B is not a strongly noncentral extension of a torsion-free $\mathscr{A}$ group.*

THEOREM 8. *$\sigma(A, B$; cyclic$)\subset R\mathcal{F}$ for A, B polycyclic-by-finite.*

This theorem is best-possible in view of Theorem 7 and Higman's example [4] of a non-Hopf group constructed as a g.f.p. of two f.g. metabelian groups with cyclic amalgamation.

It is with much pleasure that I thank my supervisor, Gilbert Baumslag, for his many suggestions, his kindness and encouragement. I wish also to express my thanks to the referee for many corrections and improvements, most particularly of Theorems 5 and 8.

Proof of Theorem 3. Let p be any prime. Define $A\cong B$ as follows:

$$A = gp\{a_1, a_2, a_3, \alpha, d \mid [a_i, a_j] = 1, a_1^\alpha = a_2, a_2^\alpha = a_3, a_3^\alpha = a_1a_2^{-3}a_3^3,$$
$$d^p = [d, \alpha] = [d, a_i] = 1; i, j = 1, 2, 3\},$$
$$B = gp\{b_1, b_2, b_3, \beta, e \mid [b_i, b_j] = 1, b_1^\beta = b_2, b_2^\beta = b_3, b_3^\beta = b_1b_2^{-3}b_3^3,$$
$$e^p = [e, \beta] = [e, b_j] = 1; i, j = 1, 2, 3\}.$$

These groups are isomorphic to the direct product of Z_p with a split extension of a free abelian rank 3 group by an infinite cycle, and are nilpotent of class 3.

Define $H<A$, $K<B$ as:

$$H = gp\{a_1, a_2^p, a_3, d\}, \qquad K = gp\{b_1^pe, b_2, b_3^p, e\}.$$

Then $H\cong K\cong Z\times Z\times Z\times Z_p$, and we identify them via the isomorphism $\varphi\colon H\to K$ given by

$$\varphi a_1 = b_1^pe, \quad \varphi a_2^p = b_2, \quad \varphi a_3 = b_3^p, \quad \varphi d = e.$$

Set $P=\{A * B; H\}$, the g.f.p. of A and B with H $(=K)$ amalgamated. Then we claim that

$$d\in \bigcap \{N\colon N \lhd P, P/N\in \mathcal{F}\}$$

which is therefore not free as d is of finite order. To this end, suppose there does exist $N \lhd P$ with $P/N\in\mathcal{F}$ and $d\notin N$. For each $w\in P$, let $|w|$ denote the order of $wN\in P/N$; then $|w|<\infty$ and clearly

$$\forall x\in P,\ |w^x| = |w|, \quad |w^p| = |w|/(p, |w|).$$

Thus, e.g. $|a_1|=|a_2|=|a_3|$; $|b_1|=|b_2|=|b_3|$.

Now $d\notin N$ so $|d|=p$. Let $n=|a_1|$; then $n=n_1p^s$, $(n_1, p)=1$. Interpret, for $q\in Z$ $[q]=\max\{0, q\}$; then $|a_2^p|=n_1p^{[s-1]}$ and, using $a_2^p=b_2$ and $|b_1|=|b_2|$, $|b_1^p|=n_1p^{[s-2]}$.

Now $|b_1^p|=|b_3^p|=|a_3|=|a_1|=n_1p^s$ and so we must have had $s=0$, or, $(n, p)=1$. But then $|b_1^pe|=np\neq n=|a_1|$ which is an impossibility. Thus $|d|=1$, or, $d\in N$ as claimed.

Proof of Theorem 4. We require the following rather technical

PROPOSITION 1. *Let $P=\{A * B; H\}$ and suppose $M \lhd A$, $N \lhd B$ with $M\cap H$ $=N\cap H$. Then*

(a) *$nm_p\{M, N\} \cong \{\ast_{\gamma\in\Gamma} G_\gamma \mid H_{\gamma\gamma'}\}$, the g.f.p. of the groups G_γ, $\gamma \in \Gamma$ with amalgamated subgroups $H_{\gamma\gamma'} = G_\gamma \cap G_{\gamma'}$, where G_γ is a conjugate in P of M or N and $H_{\gamma\gamma'}$ is a conjugate of a subgroup of $H \cap M = H \cap N$. Furthermore*:

(b) *There exists a g.f.p. $\Sigma \in \sigma(M^*, N^*)$, where M^*, N^* are subgroups of the holomorphs of M, N; and a homomorphism $\theta: nm_p\{M, N\} \to \Sigma$ such that, if $\delta: \Sigma \to K$ is any homorphism whose restrictions to the factors M^*, N^* of Σ are injective, then $\delta \circ \theta: nm_p\{M, N\} \to K$ is injective on each factor G_γ of $nm_p\{M, N\}$.*

Let us withhold the proof of this proposition till Theorem 4 has been established.

Let $P = \{A * B;\ H\} \in \sigma(A, B)$ with A, B f.g. nilpotent. Then

$$\tau H = \tau A \cap H = \tau B \cap H$$

(τG denoting the torsion portion of G) and we may apply the proposition above to $nm_p\{\tau A, \tau B\}$. Now A is f.g. nilpotent, so τA is finite and therefore its holomorph is also finite. But then the g.f.p. Σ of the proposition is $R\mathcal{F}$ as Theorem 1 is applicable. Thus we may map Σ onto a finite group G via a homomorphism δ, injective on the factors of Σ. The composed map $\psi: nm_p\{\tau A, \tau B\} \to G$ is therefore injective on the factors of $nm_p\{\tau A, \tau B\}$ and so Ker ψ is free [6]. Thus $nm_p\{\tau A, \tau B\} \in \Phi \cdot \mathcal{F} \subset R\mathcal{F}$.

Now

$$P/nm_p\{\tau A, \tau B\} \cong P^* = \{A/\tau A * B/\tau B;\ H/\tau H\},$$

but $A/\tau A$, $B/\tau B$ are torsion free and so Theorem 2 yields $P^* \in \Phi \cdot R\mathcal{F}$; thus $P \in R\mathcal{F} \cdot \Phi \cdot R\mathcal{F}$. If H is closed in A, B then $\tau A = \tau H = \tau B$ so $nm_p\{\tau A, \tau B\} = \tau H \in \mathcal{F}$, while $H/\tau H$ is closed in $A/\tau A$, $B/\tau B$ whence $P^* \in R\mathcal{F}$ (Theorem 2 again), or, $P \in \mathcal{F} \cdot R\mathcal{F}$.

It is perhaps worthwhile to state explicitly the following

Corollary. *If A, B are f.g. nilpotent,*

$$\sigma(A, B;\ \tau A \cup \tau B \subset H) \subset R\mathcal{F} \cdot R\mathcal{F}.$$

As above,

$$nm_p\{\tau A, \tau B\} = \tau H \in \mathcal{F}$$

so $P \in \mathcal{F} \cdot \Phi \cdot R\mathcal{F}$, and it suffices to show $\mathcal{F} \cdot \Phi \subset R\mathcal{F}$.

To this end, let $G \in \mathcal{F} \cdot \Phi$. Now any extension by a free group splits, so there is a free subgroup $R \leqq G$, of finite index in G. Hence R has finitely many distinct conjugates, and so the normal subgroup

$$F = \bigcap_{g \in G} R^g$$

is again of finite index in G. But $F \leqq R$, so is free; thus $G \in \Phi \cdot \mathcal{F} \subset R\mathcal{F}$.

Proof of Proposition 1. In her paper *Generalized free products with amalgamated subgroups*, II [7] and as applied to g.f.p.'s with one amalgamated subgroup, Hanna Neumann establishes the fact that every subgroup of a g.f.p. is again a g.f.p.

With $P=\{A * B; H\}$ and $G \leqq P$, a system of generators for G is constructed recursively: for each ordinal σ, a set $\Phi_\sigma = \mathscr{T}_\sigma \cup \mathscr{S}_\sigma$ is chosen, where the elements of $\mathscr{T}_\sigma$ generate a factor of G which is a subgroup of a conjugate of A or B while the elements of $\mathscr{S}_\sigma$ generate factors of some other type. The amalgamated subgroups are all contained in conjugates of H. To establish (a), we trace Neumann's construction of the Φ_σ to ensure firstly that $\mathscr{S}_\sigma$ is empty for all σ, and secondly that the factor generated by the elements of $\mathscr{T}_\sigma$ is a conjugate of M or N. Let

$$G = nm_p\{M, N\}; \qquad P = \{A * B; H\}.$$

The Φ_σ are selected as follows:

Set $\Phi_0 = G \cap H$. Assume $\Phi_{\sigma'}$ has been chosen for all ordinals $\sigma' < \sigma$ and let

$$K_\sigma = gp\{w : w \in \Phi_{\sigma'}, \sigma' < \sigma\}.$$

If $K_\sigma \neq G$, define Φ_σ as follows: let

$$l = \min\{l(w) : w \in G - K_\sigma\}.$$

(where $l(w)$ denotes the length of $w \in P$; cf. [6]). If, among the elements of length l in $G - K_\sigma$ there is an element of the form $u^{-1}tu$, $t \in A \cup B$, $u \in P$ (briefly, a *transform*) *and* in normal form as written, choose one such element and, with reference to it, set

$$\mathscr{T}_\sigma = \{u^{-1}t'u : t', t \text{ in the same factor of } P, u^{-1}t'u \in G\}$$

and

$$\mathscr{S}_\sigma = \{f : f \in G - gp\{K_\sigma, \mathscr{T}_\sigma\}, l(f) = l, (f^{-1}u^{-1}tu) \leqq l\}.$$

If there is no transform of length l in $G - K_\sigma$, set $\mathscr{T}_\sigma = \varnothing$ and choose any $f \in G - K_\sigma$ of minimal length l. Then define

$$\mathscr{S}_\sigma = \{g : g \in G - K_\sigma, l(g) = l, l(g^{-1}f) \leqq l\}.$$

In either case, $\Phi_\sigma = \mathscr{T}_\sigma \cup \mathscr{S}_\sigma$. We must show

(i) $\forall \sigma$, $\mathscr{S}_\sigma = \varnothing$.

(ii) If $u^{-1}tu \in \mathscr{T}_\sigma$, then $gp\{\mathscr{T}_\sigma\} = u^{-1}Mu$ or $u^{-1}Nu$ if $t \in A$ or B.

Now (ii) is obvious: $gp\{\mathscr{T}_\sigma\}$ is generated by conjugates of elements from A or B—for definiteness assume

$$gp\{\mathscr{T}_\sigma\} \subset u^{-1}Au, \quad \text{some } u \in P.$$

We must verify

$$u^{-1}Au \cap G = u^{-1}Mu.$$

But this is equivalent to

$$A \cap G = M$$

while

$$P^* = P/G = \{A/M * B/N; H/H \cap M\}$$

so $A \cap G - M$ is empty, or, $A \cap G \subseteq M$, while apparently $M \subseteq A \cap G$.

To establish (i), we first note that every element of length $\leqq 1$ in G is a transform (necessarily with $u=1$), so if $l \leqq 1$ then $\mathscr{T}_\sigma = A \cap G$ or $B \cap G$. In this case $f \in G$, $1 \geqq l(f) = l \geqq l(f^{-1}t)$ implies that f is in the same factor of P as t and so $f \in gp\{K_\sigma, \mathscr{T}_\sigma\}$: thus $\mathscr{S}_\sigma = \varnothing$.

Suppose $l > 1$, let $g \in G - K_\sigma$, $g = \xi_1 \xi_2 \cdots \xi_l$ with $\xi_i \in (A \cup B) - H$ and ξ_i, ξ_{i+1} from different factors of P. In the natural map $P \to P^* = P/G$ given by $w \to wG$, we have

$$1 = gG = (\xi_1 G)(\xi_2 G) \cdots (\xi_l G)$$

and so $\xi_j \in MH \cup NH$ for some j. By passing to g^{-1} if necessary we may assume $j \geqq (l+1)/2$. As $MH \cup NH = (M \cup N)H = H(M \cup N)$, we may multiply ξ_{j-1} or ξ_{j+1} by an element of H (if necessary) to achieve that $\xi_j \in M \cup N \subset G$. Put $g = \eta\xi$ with

$$\eta = \xi_1 \cdots (\xi_{j-1}\xi_{j+1}) \cdots \xi_l, \qquad \xi = \xi_j^{\xi_{j+1} \cdots \xi_l}.$$

As $\xi \in G$, also $\eta \in G$; since $l(\eta) < l$, the minimal choice of l implies that $\eta \in K_\sigma$. As $g \notin K_\sigma$, it follows that $\xi \notin K_\sigma$ and so $l(\xi) = l$: this can only happen if $j = \frac{1}{2}(l+1)$. In this case ξ as written is a transform of length l in $G - K_\sigma$, so $\mathscr{T}_\sigma$ is not empty; say $u^{-1}tu \in \mathscr{T}_\sigma$, $l(u^{-1}tu) = l$. Now $\mathscr{S}_\sigma$ consists of elements g in $G - gp\{K_\sigma, \mathscr{T}_\sigma\}$ which satisfy $l(g^{-1}u^{-1}tu) \leqq l$. This last condition visibly implies that $u = (\xi_1 \cdots \xi_{j-1})^{-1}$ and that ξ_j, t are in the same factor of P: so every g satisfying the last condition is in $gp\{K_\sigma, \mathscr{T}_\sigma\}$ and therefore $\mathscr{S}_\sigma$ is empty.

To establish (b) of our proposition, recall that the holomorph, $\mathrm{Hol}\,(K)$, of a group K is the set $K \times \mathrm{Aut}\,(K)$ with product

$$(k_1, \alpha_1)(k_2, \alpha_2) = (k_1\alpha_1(k_2), \alpha_2\alpha_1).$$

Identify K with $K \times \mathrm{id}$ and $\mathrm{Aut}\,(K)$ with $1 \times \mathrm{Aut}\,(K)$.

Let $\beta_A\colon A \to \mathrm{Aut}\,(M)$, $\beta_B\colon B \to \mathrm{Aut}\,(N)$ be defined by conjugation: $\beta_A(a)(m) = m^a$, $\beta_B(b)(n) = n^b$, for all $a \in A$, $m \in M$, $b \in B$, $n \in N$. Define

$$M^* = gp\{M, \beta_A(A)\} \leqq \mathrm{Hol}\,(M), \qquad N^* = gp\{N, \beta_B(B)\} \leqq \mathrm{Hol}\,(N)$$

and set

$$\begin{aligned} H^* &= \{(h, \beta_A(h')) : h \in H \cap M,\ h' \in H\} \\ &= \{(h, \beta_B(h')) : h \in H \cap N,\ h' \in H\}. \end{aligned}$$

Then $H^* \leqq \mathrm{Hol}\,(H)$, $H^* \leqq M^*$, N^* and so we may form

$$\Sigma = \{M^* * N^*; H^*\}.$$

Define $\theta_\gamma\colon G_\gamma \to \Sigma$ as follows: for $g_\gamma \in G_\gamma$, $g_\gamma = u^{-1}tu$ with $t \in M \cup N$, $u = a_1b_1 \cdots a_kb_k$, $a_i \in A$, $b_j \in B$, set

$$\begin{aligned} \theta_\gamma(g_\gamma) &= (1, \beta_B b_k^{-1})(1, \beta_A a_k^{-1}) \cdots (1, \beta_A a_1^{-1})(t, 1)(1, \beta_A a_1) \cdots (1, \beta_A a_k)(1, \beta_B b_k) \\ &= (1, \beta_A a_1^{-1} \cdots \beta_A a_k^{-1} \beta_B b_k^{-1})(t, 1)(1, \beta_B b_k \beta_A a_k \cdots \beta_A a_1). \end{aligned}$$

Now θ_γ is well defined: a choice appears in the representation of g_γ as the product $u^{-1}tu$ of elements coming alternately from A and B, which are only determined modulo H. However, the amalgamated subgroup H^* was designed so as to void this difficulty. Then $\theta_\gamma\colon G_\gamma \to \Sigma$ is clearly monomorphic. It is also clear that

$$\theta_\gamma|_{H\gamma\gamma'} = \theta_{\gamma'}|_{H\gamma'\gamma},$$

by choice of H^*. Thus we may extend the θ_γ to an epimorphism $\theta\colon nm_p\{M, N\} \to \Sigma$ with $\theta|_{G_\gamma} = \theta_\gamma$. Since $\theta(G_\gamma)$ is in a conjugate of M^* or N^* in Σ, any map injective on M^* and N^* is also injective on $\theta(G_\gamma)$.

Proof of Theorem 5. Let

$$1 \to M \to A \to S \to 1, \qquad 1 \to N \to B \to T \to 1$$

be split, strongly noncentral with M, N torsion-free f.g. abelian.

View M as a Z-module, and form the Q-module $M^+ = M \otimes_Z Q$. Then $M \cong M \otimes_Z Z < M^+$ and we shall regard $M < M^+$. The action of A on M affords a representation of the finite group $A/\mathscr{Z}_A(M)$ as a group of linear transformations over Z which we view as a representation over Q.

Choose $m \in M$, $a \in A$ so that $gp\{m, m^a\}$ is noncyclic, and let K_1 be the normal closure of m in A. By Maschke's Theorem, $K_1 \otimes_Z Q$ has a complement, say M', in $M \otimes_Z Q$; put $K_2 = M' \cap M$. Then $K_2 \lhd A$, $K_1 \cap K_2 = 1$, and $[M{:}K_1K_2]$ is finite. Choose $n \in N$, $b \in B$ so that $gp\{n, n^b\}$ is noncyclic and construct subgroups L_1, L_2 of N similarly. Let p, q be distinct primes with p congruent to 1 mod $[M{:}K_1K_2] \times [N{:}L_1L_2]$. Define $H \leqq A$, $J \leqq B$ as follows:

$$H = gp\{m, m^a\}, \qquad J = gp\{n^p, n^{bq}\}.$$

Then $H \cong J$ with isomorphism given by

$$m \to n^p, \qquad m^a \to n^{bq}.$$

Identify H with J accordingly and form

$$P = \{A * B; H\,(=J)\}.$$

Then P is the required non-Hopf group: we establish this by exhibiting an epimorphism $\theta\colon P \to P$ with nontrivial kernel.

To this end, define $\psi\colon K_1K_2 \to K_1K_2$ by

$$\psi|_{K_1}(k) = k^p, \qquad \psi|_{K_2} = id_{K_2}.$$

We claim ψ has a (unique) extension $\bar{\psi}\colon M \to M$. Choose a basis $x_1, \ldots, x_l$ for M such that $x_1^{e_1}, \ldots, x_j^{e_j}$ form a basis for K_1 and $x_{j+1}^{e_{j+1}}w_{j+1}, \ldots, x_l^{e_l}w_l$ form a basis for K_2 where $w_i = w_i(x_1, \ldots, x_j)$. For each i, $j+1 \leqq i \leqq l$, there exists $v_i = v_i(x_1, \ldots, x_j)$ such that $w_i^{1-p} = v_i^{e_i}$, for $e_i \mid [M{:}K_1K_2]$ and $p \equiv 1 \bmod [M{:}K_1K_2][N{:}L_1L_2]$. Define $\bar{\psi}\colon M \to M$ by setting

$$\begin{aligned} \bar{\psi}(x_i) &= x_i^p, \quad i = 1, \ldots, j, \\ &= x_iv_i, \quad i = j+1, \ldots, l, \end{aligned}$$

and extending linearly. One verifies easily that $\bar{\psi}|_{K_1K_2}=\psi$. Now $\psi\colon K_1K_2\to K_1K_2$ is compatible with the action of S as K_1, K_2 are normal subgroups of A and $(k^a)^p=(k^p)^a$ for all $a\in A$, $k\in K_1$. But this implies that ψ is also S-compatible as $[M:K_1K_2]<\infty$ and M is torsion-free. Thus the splitting of the extension A allows us to assert the existence of a homomorphism $\theta_A\colon A\to A$ rendering the diagram

$$\begin{tikzcd}
1 \arrow[r] & M \arrow[r] \arrow[d, "\psi"] & A \arrow[r] \arrow[d, "\theta_A"] & S \arrow[r] \arrow[d, "id_S"] & 1 \\
1 \arrow[r] & M \arrow[r] & A \arrow[r] & S \arrow[r] & 1
\end{tikzcd}$$

commutative. Construct $\theta_B\colon B\to B$ similarly. It is clear that $\theta_A|_H=\theta_B|_H$ and so we may simultaneously extend them to an endomorphism $\theta\colon P\to P$.

Firstly, θ is an epimorphism; we need only verify that M, $N\subset\operatorname{Im}\theta$ as θ acts as the identity elsewhere. Now $m\in\operatorname{Im}\theta$ for $m=n^p=\theta(n)$. Furthermore $(p,q)=1$ so there exist u, $v\in Z$ with $pu+qv=1$. Thus $n=\theta(n)^u n^{qbvb^{-1}}=\theta(n)m^{avb^{-1}}$ and m, $a\in\operatorname{Im}\theta$ so $n\in\operatorname{Im}\theta$ as well. Thus $K_1=nm_A\{m\}$, $L_1=nm_B\{n\}\subset\operatorname{Im}\theta$. But $|\tau(M/K_1)|$, $|\tau(N/L_1)|$ are prime to p, and θ acts as the identity on M/K_1 modulo $\tau(M/K_1)$, N/L_1 modulo $\tau(N/L_1)$.

Let $w=[n,ab^{-1}]^p n^{p-q}$. Then $w\in\operatorname{Ker}\theta$ but $w\neq 1$ since n, $n^{p-q}\in B-H$ and $a\in A-H$.

Proof of Theorem 6. In all of the following, the aim is always to reduce the problem to the case in which the amalgamated subgroup is finite and Baumslag's Theorem 1 is applicable. Formally, the situation we shall obtain is that described by the hypotheses of the following proposition, essentially due to Baumslag [1]:

PROPOSITION 2. *Let $P=\{A*B;H\}$ and assume there exist equally-indexed families $\{A_n\}_{n\in Z^+}$, $\{B_n\}_{n\in Z^+}$ of nested normal subgroups of A, B (i.e. filtrations of A, B) satisfying*

(i) $\forall n\in Z^+$, $H\cap A_n=H\cap B_n$,

(ii) $\forall n\in Z^+$, $H/H\cap A_n\in\mathscr{F}$; A/A_n, $B/B_n\in R\mathscr{F}$,

(iii) $\bigcap_{n\in Z^+}A_n=1=\bigcap_{n\in Z^+}B_n$,

(iv) $\bigcap_{n\in Z^+}HA_n=H=\bigcap_{n\in Z^+}HB_n$.

Then $P\in R\mathscr{F}$.

Now (i) establishes the existence of epimorphisms

$$\theta_n\colon P\to P_n=\{A/A_n*B/B_n;\ H/H\cap A_n=H/H\cap B_n\}$$

extending the canonical projections $A\to A/A_n$, $B\to B/B_n$ and $P_n\in R\mathscr{F}$ by (ii) and Theorem 1. But any $w\in P$ is a finite product of elements coming alternately from A and B, each of which may be excluded from $\operatorname{Ker}\theta_n$ for some n by (iii). Moreover, we can ensure that $l_P(w)=l_{P_n}(\theta_n w)$ for n sufficiently large, as the image of any element of $A-H$ or $B-H$ lies in $A/A_n-HA_n/A_n$ or $B/B_n-HB_n/B_n$ for n large enough by (iv). Thus $P\in R(R\mathscr{F})=R\mathscr{F}$.

Let $P=\{A * B; H\} \in \sigma(A, B)$ and let $K \lhd A$, $L \lhd B$ be finite with A/K, B/L torsion-free abelian. Now HK, $HL \lhd A$, B respectively. Since HK, $HL \in R\mathscr{F}$ there exists an integer r for which

$$(HK)^r \cap K = 1 = (HL)^r \cap L.$$

Since $K \geqq [A, A]$, $L \geqq [B, B]$; $(HK)^r$, $(HL)^r$ are central. Now $(HK)^r \geqq H^r$, $(HL)^r \geqq H^r$ hence $H^r \lhd A, B$. Thus $\{H^{rq^n}\}_{n \in Z^+}$ forms a filtration of both A and B of the required type: conditions (i), (ii) and (iv) are immediate, while (iii) follows from the fact that no $\neq 1$ element in any f.g. torsion-free abelian group is of infinite height.

Proof of Theorem 7. Suppose that $A \in \mathscr{A} \cdot \mathscr{F}$ is *not* a strongly noncentral extension of a torsion-free abelian group. Then there exists $M \lhd A$ with $M \in \mathscr{A}$, $A/M \in \mathscr{F}$ and $gp\{m, m^a\}$ cyclic for every $m \in M$, $a \in A$. We may replace M by $M^{|\tau(M)|}$ to ensure that M is torsion-free. Note that, for $m \in M$ and root-free, $m^a = m^{\pm 1}$; so this is the case for any element of a basis for M.

Now for any subgroup N of M there is a basis $m_1, \ldots, m_K$ of M so that N has basis $m_1^{\varepsilon_1}, \ldots, m_K^{\varepsilon_K}$, $\varepsilon_i \geqq 0$ and integral. Since the m_i are root free, for all $a \in A$ we have

$$(m_i^{\varepsilon_i})^a = m_i^{\pm \varepsilon_i},$$

thus $N^a = N$ as N is a subgroup; i.e. any subgroup of M is normal in A.

Now let $A, B \in \mathscr{A} \cdot \mathscr{F}$ and $H \leqq B, A$ with A not strongly noncentral. Let $N \lhd B$ with $N \in \mathscr{A}$, $B/N \in \mathscr{F}$. There exists an integer r for which $A^r \leqq M$, $B^r \leqq N$ and A^r, B^r are torsion-free. Let $t = |\tau(B^r/H^r)|$, and, for any integer $s > 1$ define, for each $n \in Z^+$

$$B_n = B^{rts^n}, \qquad A_n = H \cap B_n.$$

Then $B^{rt} \cap H \leqq H^r$ by choice of t, for

$$H \cap B^{rt} \leqq H^r \cap B^r \leqq H^r.$$

Hence $A_n \leqq H^r \leqq A^r \leqq M$, by choice of r; so $A_n \lhd A$. Now observe that

$$\bigcap_{n \in Z^+} B_n = 1 = \bigcap_{n \in Z^+} A_n, \quad H \cap A_n = H \cap B_n, \quad H/H \cap B_n \in \mathscr{F}.$$

Furthermore $\mathscr{A} \cdot \mathscr{F} \subset R\mathscr{F}$ and the class of $\mathscr{A} \cdot \mathscr{F}$ groups is image-closed, so

$$A/A_n, B/B_n \in R\mathscr{F}.$$

Since $A_n \leqq H$, apparently

$$\bigcap_{n \in Z^+} HA_n = H$$

and we may apply Proposition 2 once we verify

$$\bigcap_{n \in Z^+} HB_n = H;$$

we must show $\bigcap_{n \in Z^+} HB_n \leqq H$ for the reverse inclusion is automatic. Now $H \cap B^{rt}$

is a direct factor of B^{rt} as $B^{rt}/H \cap B^{rt}$ is torsion-free (our selection of t) and B^{rt} is a torsion-free $\mathscr{A}$ group. Thus $H \cap B^{rt}$ is complemented by a subgroup $K \leqq B^{rt}$:

$$B^{rt} = (H \cap B^{rt})K; \qquad K \cap H \cap B^{rt} = 1$$

whence, for all $n \in Z^+$, $HB^{rts^n} = HK^{s^n}$ and $K \cap H = 1$ as $K \leqq B^{rt}$. Suppose $b \in \bigcap_{n \in Z^+} HB_n$; i.e. $b \in HK^{s^n}$ for all $N \in Z^+$. Thus

$$b = h_n k_n^{s^n}, \quad h_n \in H, k_n \in K.$$

But $K \cap H = 1$ so $h_1 = h_n$ for all n, whence

$$h_1^{-1} b = k_n^{s^n} \in K^{s^n}.$$

Since K is f.g. torsion-free abelian, $\bigcap_{n \in Z^+} K^{s^n} = 1$ or $b = h_1 \in H$ as required.

To prove the remaining part of Theorem 7, let $A \in \mathscr{A} \cdot \mathscr{F}$ be a strongly noncentral extension of a torsion-free $\mathscr{A}$-group. Thus there exists $m \in A$ whose normal closure is a torsion-free $\mathscr{A}$-group of rank at least two. Choose $a \in A$ such that $gp\{m, m^a\}$ is noncyclic. There is a least integer $K > 0$ with $[a^K, m] = 1$. Set $C = gp\{a, m\} \leqq A$. For $B \in \mathscr{A} \cdot \mathscr{F}$ any other strongly noncentral extension of a torsion-free $\mathscr{A}$ group, form $D = gp\{b, n\}$ similarly. To show $\sigma(C, D) \not\subseteq R\mathscr{F}$ is sufficient as every element of $\sigma(C, D)$ is a subgroup of some element of $\sigma(A, B)$. In fact, $\sigma(C, D)$ contains a non-Hopf group, and this is the content of the following proposition.

PROPOSITION 3. *Let $C, D \in \mathscr{A} \cdot \mathscr{F}$, with*

$$1 \to M \to C \to S \to 1, \qquad 1 \to N \to D \to T \to 1$$

such that M, N are noncyclic torsion-free $\mathscr{A}$-groups, S, T are cyclic, M and N are one generator S and T modules respectively and $[S^K, M] = [T^L, N] = 1$ for some positive integers K, L. Then $\sigma(C, D)$ contains a non-Hopf group.

Let $m \in M$, $n \in N$ be elements whose normal closures in C, D generate M, N. Choose $c \in C$, $d \in D$ such that cM, dN generate S, T (regarding $S = C/M$, $T = D/N$). Assuming K, L integers such that $[S^K, M] = [T^L, N] = 1$, $c^K \in M$ or $gp\{c\} \cap M = 1$ and $d^L \in N$ or $gp\{d\} \cap N = 1$, choose p to be any prime of the form

$$1 + KLr, \qquad r \in Z^+.$$

(As is well known, there are infinitely many such for the numbers $1 + KLn$, $n \in Z^+$ form an arithmetic sequence with $(1, KL) = 1$.) Let $q > 1$ be any number prime to p. Then

$$H = \{m, m^c), \qquad H^{\times} = \{n^p, n^{qd}\}$$

are free abelian of rank 2 and may be identified via the isomorphism $\varphi: H \to H^{\times}$ given by

$$\varphi m = n^p \qquad \varphi m^c = n^{qd}.$$

Set $P = \{C * D;\ H (= H^{\times})\} \in \sigma(C, D)$ and we claim P is non-Hopf; this is established, as before, by exhibiting an ependomorphism of P with nontrivial kernel: Suppose $c^K \in M$. Then each element of C may be written in the form

$$c^\varepsilon\mu, \quad 0 \leqq \varepsilon < K, \mu \in M.$$

Define $\theta\colon C \to C$ by

$$\theta(c^\varepsilon\mu) = c^{p\varepsilon}\mu^p.$$

If $c^K \notin M$, $gp\{c\} \cap M = 1$ and every element of C may be written uniquely in the form

$$c^\varepsilon\mu, \quad \varepsilon \in Z^+, \mu \in M.$$

Then define

$$\theta(c^\varepsilon\mu) = c^\varepsilon\mu^p.$$

In either case θ is a well-defined homomorphism.

With $\rho\colon D \to D$ defined similarly, it is clear that ρ and θ agree on H and so may simultaneously be extended to an endomorphism $\psi\colon P \to P$. Now $(p, K) = 1$ so $\mu \in M \cap \operatorname{Im}\psi$. But therefore $M \subset \operatorname{Im}\psi$ as $m = \psi(n)$ while $M = gp\{m^{c^\varepsilon} : \varepsilon \in Z^+\}$. Furthermore

$$c = c^p c^{-KLr}$$

so in the situation $c^K \in M$, we have $\psi(c) = c^p$ whence $c \in \operatorname{Im}\psi$; while if $c^K \notin M$ then $c = \psi(c)$, therefore $C \subset \operatorname{Im}\psi$. Similarly, $D \subset \operatorname{Im}\psi$ provided $n \in \operatorname{Im}\psi$; but $(q, p) = 1$ so there exist $u, v \in Z^+$ with $qu + pv = 1$. Thus

$$n = n^{qu+pv} = \psi(n)^v(n^{qd})^{ud^{-1}} = \psi(n)^v m^{cud^{-1}}.$$

Now $m, c \in \operatorname{Im}\psi$, $m^{cu} \in N \cap \operatorname{Im}\psi$ and therefore $(m^{cu})^{d^{-1}} \in \operatorname{Im}\psi$. Thus also $D \subset \operatorname{Im}\psi$ and ψ is an epimorphism. However, with $w = [n, cd^{-1}]^p n^{p-q}$ we find $w \in \operatorname{Ker}\psi$, $w \neq 1$ as $n, n^d \in D - H$ while $c \in C - H$.

Proof of Theorem 8. Let us observe that, for A any polycyclic-by-finite group, there exists a sequence of integers $\{r_n\}_{n\in Z^+}$ with

$$r_n | r_{n+1}, \quad \bigcap_{n\in Z^+} A^{r_n} = 1$$

whence also

$$\bigcap_{n\in Z^+} A^{r_n s_n} = 1$$

for any sequence $\{s_n\}_{n\in Z^+}$ of integers. This may be established directly or by an easy application of a result of Learner [5], where it is in fact shown that we may choose $r_n = k^n$ for some $k \in Z^+$.

Let $P = \{A * B; H\} \in \sigma(A, B; \text{cyclic})$. Utilizing Theorem 1 as usual, we may assume that H is infinite: $H = gp\{x\}$. As A, B are polycyclic-by-finite, choose series

$$A = A_1 \rhd A_2 \rhd \cdots \rhd A_K \rhd 1, \qquad B = B_1 \rhd B_2 \rhd \cdots \rhd B_L \rhd 1$$

with $A_i \lhd A$, $B_j \lhd B$ for $1 \leqq i \leqq K$, $1 \leqq j \leqq L$, whose factors are either finite or torsion-free $\mathscr{A}$. Choose i, j minimal for which

$$H \cap A_{i+1} = 1 = H \cap B_{j+1}.$$

Then $H \cap A_i \neq 1 \neq H \cap B_j$, or

$$H \cap A_i = gp\{x^r\}, \qquad H \cap B_j = gp\{x^s\}, \quad r, s > 0;$$

and A_i/A_{i+1}, B_j/B_{j+1} are infinite, hence torsion-free $\mathscr{A}$. Thus there exist maximal integers u, v with

$$x^{rs}A_{i+1} \in (A_i/A_{i+1})^u, \qquad x^{rs}B_{j+1} \in (B_j/B_{j+1})^v,$$

that is,

$$H \cap A_i^u A_{i+1} = gp\{x^{rs}\} = H \cap B_j^v B_{j+1}.$$

Now $gp\{x^{rs}A_{i+1}\}$ is a direct factor of $(A_i/A_{i+1})^u$, so that

$$x^{rst}A_{i+1} \in (A_i/A_{i+1})^{u\rho} \quad \text{if and only if} \quad \rho | t.$$

As

$$(A_i/A_{i+1})^{u\rho} = (A_i^u A_{i+1})^\rho A_{i+1}/A_{i+1}$$

and the situation is symmetric, this means that

$$H \cap (A_i^u A_{i+1})^\rho A_{i+1} = gp\{x^{rs\rho}\} = H \cap (B_j^v B_{j+1})^\rho B_{j+1}$$

for every ρ in Z^+. On account of $x^{rs} \in A_i^u A_{i+1} \cap B_j^v B_{j+1}$, in fact

$$H \cap (A_i^u A_{i+1})^\rho = gp\{x^{rs\rho}\} = H \cap (B_j^v B_{j+1})^\rho.$$

Since $A_i^u A_{i+1}$, $B_j^v B_{j+1}$ are polycyclic-by-finite, there exists a sequence $\{\rho_n\}_{n \in Z^+}$ of positive integers, with $\rho_n | \rho_{n+1}$ for all n, such that

$$\bigcap_{n \in Z^+} (A_i^u A_{i+1})^{\rho_n} = 1 = \bigcap_{n \in Z^+} (B_j^v B_{j+1})^{\rho_n}.$$

Set

$$C_n = (A_i^u A_{i+1})^{\rho_n}, \qquad D_n = (B_j^v B_{j+1})^{\rho_n}.$$

Then $\{C_n\}_{n \in Z^+}$, $\{D_n\}_{n \in Z^+}$ form nested filtrations of A, B; the proof will be completed by showing that these satisfy the hypotheses of Proposition 2. The nontrivial part is to check that, e.g.,

$$\bigcap_{n \in Z^+} C_n H = H.$$

From above, $H \cap C_n A_{i+1} = gp\{x^{rs\rho_n}\} = H \cap C_n$. Since $HA_i^u A_{i+1}/A_i^u A_{i+1}$ is finite, there exists to each g in $\bigcap_{n \in Z^+} C_n H$ an element h in H such that

$$hg \in \bigcap_{n \in Z^+} C_n(H \cap A_i^u A_{i+1}).$$

As in a step of the proof of Theorem 7, it can be deduced that

$$\bigcap_{n \in Z^+} C_n(H \cap A_i^u A_{i+1})A_{i+1}/A_{i+1} = (H \cap A_i^u A_{i+1})A_{i+1}/A_{i+1},$$

that is,

$$\bigcap_{n \in Z^+} C_n(H \cap A_i^u A_{i+1})A_{i+1} = (H \cap A_i^u A_{i+1})A_{i+1};$$

thus $hg \in (H \cap A_i^u A_{i+1})A_{i+1}$. This proves that $\bigcap_{n \in Z^+} C_n H \leqq HA_{i+1}$. Obviously, $H \leqq \bigcap_{n \in Z^+} C_n H$; it remains to establish that $\bigcap_{n \in Z^+} C_n H \cap A_{i+1} = 1$. This certainly holds if, for all n in Z^+, $C_n H \cap A_{i+1} \leqq C_n$. In fact, more is true; namely, $C_n H \cap C_n A_{i+1} \leqq C_n$. The converse inclusion being obvious, this last statement follows from the fact that the indices of $C_n H \cap C_n A_{i+1}$ and C_n in $C_n H$ are equal and finite. Indeed, as $H \cap C_n = H \cap C_n A_{i+1} = gp\{x^{rs\rho_n}\}$,

$$\begin{aligned} C_n H / C_n H \cap C_n A_{i+1} &\cong C_n H A_{i+1} / C_n A_{i+1} \cong H / H \cap C_n A_{i+1} \\ &= H / H \cap C_n \cong C_n H / C_n \end{aligned}$$

shows that each index is $rs\rho_n$. This completes the proof of Theorem 8.

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Columbia University,
New York, New York